OVERVIEW:

1. Introduction to Abstract Functions
2. Average Rate of Change
3. Instantaneous Rate of Change
4. Algebra Bootcamp for Basic Derivatives
5. Basic Derivatives
6. Derivatives of Trigonometric Functions
7. Chain Rule and Implicit Differentiation
8. Related Rates
9. Logarithmic Differentiation
10. The Shape of a Graph
11. Optimization Problems
12. Anti-derivatives
13. Riemann Sums
14. Basic Integration
15. The Fundamental Theorem of Calculus (Part II)
16. Advanced Integration and Average Height
17. Rectilinear Motion
18. Area Between Curves, Volumes of Revolution, Arc Length

OUTLINE:

Notes: Each class is 50 minutes and meets 4 times per week.
Time frames are approximate and include assessments.
Asterisks (*) indicate optional topics.

Quarter 1

UNIT 1: INTRODUCTION TO ABSTRACT FUNCTIONS

1. Given a graph of a basic original function (which is only constant or linear), students *algebraically* come up with the equation for the area function associated with it. In other
words, students can algebraically find the equation for the area function associated with a given constant or linear function.

2. Given a graph of an original function (which may consist of constant and linear parts), students can draw (with reasonable accuracy) the corresponding area function.

3. Students can explain why a linear function is the area function of a constant function; students can explain why a quadratic function is the area function of a linear function.

4. Students, given an area function, can make conclusions about the original function associated with it.

UNIT 2: AVERAGE RATE OF CHANGE

1. Determine the equation of a line given two points, or a point and a slope, or a graph of a line; Rationalize the numerator of expressions like $\frac{\sqrt{x+h} - \sqrt{x}}{h}$ and understand why the process you go through doesn’t change the value of the expression.

2. Articulate what information precisely an average rate of change can provide, what it can’t provide, and when it acts as a good approximation for what is happening to the function. Similarly, articulate what information precisely an instantaneous rate of change can provide, what it can’t provide, and when it acts as a good approximation for what is happening to the function.

3. Find the average rate of change over an interval given a function or a graph of a function. Be able to articulate what the average rate of change means conceptually.

4. Approximate using a series of two points close to each other the instantaneous rate of a change for a point, given a function or a graph of a function; explain clearly why this procedure gives an approximation of the true instantaneous rate of change.

5. Clearly express what is happening to an object given a position versus time graph; sketch a velocity versus time graph given a position versus time graph

UNIT 3: INSTANTANEOUS RATE OF CHANGE

1. Clearly express what is happening to an object given a position versus time graph; be able to sketch an instantaneous rate of change graph given an original graph (in other words, a velocity versus time graph given a position versus time graph)

2. Be able to sketch tangent lines to functions at given values and be able to appropriately use this line to estimate the instantaneous rate of change at these given values; be able to compare the instantaneous rate of change of a function at various points using the tangent line (example: the instantaneous rate of change at point A is [greater than/less than/equal to] the instantaneous rate of change at point B)

3. Find the instantaneous rate of change exactly for simple polynomials at a given point
4. Be able to explain the meaning of \( h \) and how limits help us find the instantaneous rate of change; be sure to be able to articulate the mathematical reason for each step when finding the instantaneous rate of change exactly (algebraically)

**UNIT 4: ALGEBRA BOOTCAMP FOR BASIC DERIVATIVES**

1. Determine the equation of a line given two points, or a point and a slope, or a graph of a line.
2. Find the average rate of change over an interval given a function or a graph of a function.
3. Approximate using two points close to each other the instantaneous rate of change for a point, given a function or a graph of a function.
4. Explain why the slope between two points close together gives an approximation of the true instantaneous rate of change.
5. Clearly express a graphical and algebraic interpretation for average and instantaneous rate of change.
6. Clearly express what is happening to an object given a position versus time graph.
7. Sketch a velocity versus time graph given a position versus time graph.
8. Rationalize the numerator of expressions like \( \frac{\sqrt{x+h}-\sqrt{x}}{h} \).
9. Expand the expression \((x + h)^n\) using the binomial theorem.

**Quarter 2**

**UNIT 5: BASIC DERIVATIVES**

1. Write, and be able to explain clearly the graphical interpretation of, the formal definition of the derivative.
2. Apply the formal definition of the derivative to simple polynomials and to simple square root functions (e.g. \( f(x) = x^3 + 2x - 1 \) and \( y = \sqrt{2 - x} \)).
3. Use Wolfram|Alpha to apply the formal definition of the derivative to high degree monomials (like \( x^{10} \)) to explain the power rule for derivatives.
4. Apply the power rule for derivatives.
5. Determine the derivative of exponential functions.
6. Apply the sum, difference, product, and quotient rules for derivatives.
7. Graphically understand the relationship between a function and its derivative.
8. Determine the equation of the tangent line to a function at a given point.
9. Determine where a function has a vertical asymptote using derivatives.
10. Use knowledge of differentiating to find the second derivative of a function.
UNIT 6: ALGEBRA BOOTCAMP FOR DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. Evaluate $\sin(\theta), \cos(\theta)$, and $\tan(\theta)$ at special angles, without a calculator.
2. Know and apply the sum of angles formulae for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.
3. Accurately graph $\sin(\theta)$ and $\cos(\theta)$ on $[-2\pi, 2\pi]$.
4. Prove that $\frac{d}{dx}\sin(\theta) = \cos(\theta)$ and that $\frac{d}{dx}\cos(\theta) = -\sin(\theta)$.
5. Evaluate the composition of a trigonometric function with an inverse trigonometric function, given a right triangle with two sides and an angle labeled (e.g. $\tan(\sin^{-1}(\theta))$).
6. Apply the sum, difference, product, and quotient rule with trigonometric functions.

Quarter 3

UNIT 7: CHAIN RULE AND IMPLICIT DIFFERENTIATION

1. Apply the chain rule in basic derivative problems.
2. Apply the chain rule in derivative problems involving the sum, difference, product, and quotient rules.
3. Determine the derivative of functions which require the use of the chain rule more than once.
4. Derive the quotient rule by finding the derivative of $[f(x)g(x)]^{-1}$ using the product and chain rules.
5. Use implicit differentiation to find the derivative of a function in an implicit equation (e.g. a relation like $x^2 + y^2 = 16$).
6. Use implicit differentiation to find the equation of a tangent line (or tangent lines) to a function at a given $x$ value.
7. Explain clearly the action of "differentiating" something with respect to a variable and how that action relates to the chain rule; for example, why $\frac{d}{dx}[x^5] = 5x^4$ but $\frac{d}{dt}[x^5] = 5x^4 \frac{dx}{dt}$.
8. Use unit right triangles, knowledge of inverse functions, and implicit differentiation to determine the derivative of the three basic inverse trigonometric functions, $\sin^{-1}(x), \cos^{-1}(x)$, and $\tan^{-1}(x)$.
9. Use knowledge of inverse functions, the exponential function, and implicit differentiation to determine the derivative of the natural logarithm function, $\ln(x)$.

UNIT 8: RELATED RATES

1. Draw a picture representing a given situation in a related rates problem.
2. Assign values and variables to the known and unknown quantities in a related rates problem.
3. Determine the relevant main equation relating the known and unknown quantities in a related rates problem.
4. Use implicit differentiation on the relevant main equation to solve a related rates problem.

UNIT 9: LOGARITHMIC DIFFERENTIATION
1. Use logarithm rules to “break apart” a single logarithmic expression into the sum and difference of more logarithms.
2. Use logarithmic differentiation (and the chain rule) to find the derivative of functions involving a lot of multiplication and division (e.g. \( y = \frac{3x^2e^x}{5\sin(x)} \)).
3. Explain clearly how one could use logarithmic differentiation on functions involving a sum or difference of a constant (e.g. \( y = \frac{3x^2e^x}{5\sin(x)} + 1 \) [note: it would also work for non-constants].

Quarter 4

UNIT 10: THE SHAPE OF A GRAPH
1. Algebraically calculate the zeros of basic and more complicated algebraic functions including rational equations, including equations involving trigonometry, the natural logarithm and exponential functions.
2. Use derivatives to determine where a function is increasing and decreasing.
3. Use both the first and second derivatives tests to identify relative maxima, relative minima, and inflection points.
4. Determine the absolute maxima and minima of a function on a given finite or infinite interval.
5. Explain clearly why the zeros of the first derivative are not all relative maxima and minima.
6. Use second derivatives to determine where a function is concave up and concave down.
7. Determine where a function is increasing, decreasing, concave up, concave down, relative maxima, relative minima, and inflection points from a given graph of \( f(x) \).
8. Determine where a function is increasing, decreasing, concave up, concave down, relative maxima, relative minima, and inflection points from a given graph of \( f'(x) \).

UNIT 11: OPTIMIZATION PROBLEMS
1. Draw a picture representing a given situation in an optimization problem.
2. Assign values and variables to the known and unknown quantities in an optimization problem.
3. Determine the relevant main equation and the relevant helper equation for an optimization problem.
4. Use implicit differentiation on the main equation, and apply the helper equation, to solve an optimization problem.

UNIT 12: ANTIDERIVATIVES

1. Determine basic antiderivatives using intuition and the guess and check method.
2. Discover and apply basic antiderivative rules for polynomials, trigonometric functions, and exponential functions.
3. Determine antiderivatives using logic, guess and check, and the backwards chain rule.

UNIT 13: RIEMANN SUMS

1. Estimate areas of enclosed figures using only geometric arguments.
2. Estimate areas under curves using $n$ left-handed and right-handed Riemann sums.
3. Explain clearly how Riemann sums act as approximations to the true area under a curve, and how the number of rectangles chosen affects the accuracy of how close the sum is to the true area.
4. Draw the rectangles of a Riemann sum given the function, the number of intervals, and whether the Riemann sum is left- or right-handed.
5. Enter a Riemann sum program into the graphing calculator, and conceptually understand each line of the program.
6. Explain clearly why a Riemann sum can be negative if it is an approximation to the area under a curve.

UNIT 14: BASIC INTEGRATION

1. Use and understand integral notation as shorthand for “antiderivative”.
2. Use and understand integral notation as “signed area under curve” – relating the notation to an infinite Riemann sum.
3. Properly apply integral notation when showing work.
4. Use the graphing calculator’s fnInt and integration tool to estimate the area under curves.
5. Evaluate basic integrals with $u$ -substitution (e.g. $\int x(x^2 + 1)^2 dx$).
6. Evaluate integrals using basic geometry.
7. Explain clearly the role of the “$+C$” when evaluating indefinite integrals.
UNIT 15:  THE FUNDAMENTAL THEOREM OF CALCULUS (PART II)

1. Understand and clearly explain the statement of the Fundamental Theorem of Calculus, Part II (focusing on the meaning of the variable in the limits of integration) graphically.
2. Apply the Fundamental Theorem of Calculus, Part II to evaluate the derivative of integral functions.
3. Explain clearly why the constant (lower) limit in the Fundamental Theorem of Calculus does not matter, in other words why \( \frac{d}{dx} \int_{a}^{x} \sin(t^2) \, dt = \frac{d}{dx} \int_{8}^{x} \sin(t^2) \, dt \).
4. Sketch \( \int_{a}^{x} f(t) \, dt \) appropriately given a graph of \( f(t) \), paying attention to when \( \int_{a}^{x} f(t) \, dt \) is increasing, decreasing, concave up, and concave down [note: apply knowledge of how derivatives relate to the shape of a graph].

UNIT 16:  ADVANCED INTEGRATION AND AVERAGE HEIGHT

1. Determine that \( \int_{1}^{x} \frac{1}{x} \, dx = \ln |x| + C \) and apply that to simple integrals.
2. Evaluate complicated integrals with \( u \)-substitution (e.g. \( \int x(x+1)^{1/3} \, dx \) and \( \int \frac{\cos(x)}{1+\sin(x)} \, dx \)).
3. Evaluate definite and indefinite integrals involving a mixture of elements, including trigonometry, natural logarithms, exponential functions, etc.
4. Derive the integrals that are equivalent to the basic inverse trigonometric functions (i.e. \( \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + C, \int \frac{1}{\sqrt{1-x^2}} \, dx = \cos^{-1}(x) + C \) and \( \int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + C \)).
5. Apply knowledge of the inverse trigonometric integrals with \( u \)-substitution to evaluate similar-looking integrals (e.g. \( \int \frac{3}{\sqrt{2-18x^2}} \, dx \)).
6. Determine the average height of a function.
7. Explain clearly where the formula for the average height (average value) of a function comes from.
UNIT 17: RECTILINEAR MOTION

1. Given the position function of an object, describe the motion of the object in words.
2. Given the position, velocity, or acceleration of an object (and some initial conditions), determine the remaining two features of the object.
3. Explain clearly the conceptual difference between distance and displacement with regards to rectilinear motion.
4. Calculate the distance and displacement of an object in motion, given the position or velocity function of the object.
5. Apply understanding of integration, signed area, distance and displacement to answer questions about an object’s motion (note: these types of questions are AP free response questions on rectilinear motion).

UNIT 18: AREA BETWEEN CURVES, VOLUME OF REVOLUTION, AND ARCLENGTH

1. Calculate the area between two curves, both on a given interval or by determining points of intersection to provide the interval.
2. Explain clearly, using Riemann Sums, where the formula for the volume of a curve revolved about the \(x\)-axis comes from.
3. Calculate volumes of revolution for a given function over a given interval.
4. Explain clearly, using the Pythagorean Theorem, where the formula for arc length of a curve comes from.
5. Calculate the arc length of a curve over a given interval (using \(fnInt\)).